CHALMERS
UNIVERSITY OF TECHNOLOGY

Automorphic string amplitudes

Henrik Gustafsson

Potsdam 2016

hgustafsson.se

Based on

Small automorphic representations and degenerate Whittaker vectors
HG, Axel Kleinschmidt, Daniel Persson
`arXiv:1412.5625 [math.NT]` [GKP14]
Journal of Number Theory 166 (Sep, 2016) 344–399

Eisenstein series and automorphic representations
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Upcoming work with
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$SL(n)$ E_6, E_7, E_8

Outline

Compute Fourier coefficients of automorphic forms to capture information about non-perturbative effects such as instantons and black holes

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  4-graviton | Derivative expansion | U-duality | SUSY constraints

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  Parabolic subgroups | Limits of string theory | Adelic framework

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Motivation

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- Hecke eigenvalues
- Point counts of elliptic curves
- Langlands program
  L-functions | The Langlands–Shahidi method

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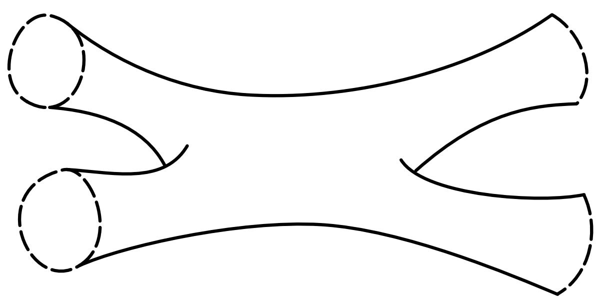
String theory

Toroidal compactifications of type IIB string theory

4-graviton scattering amplitudes



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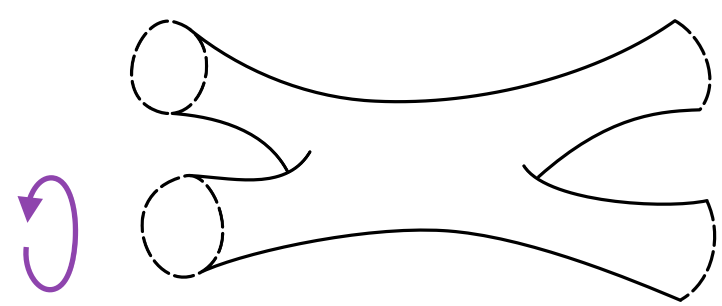
$\alpha' = \ell_{\rm s}^2$



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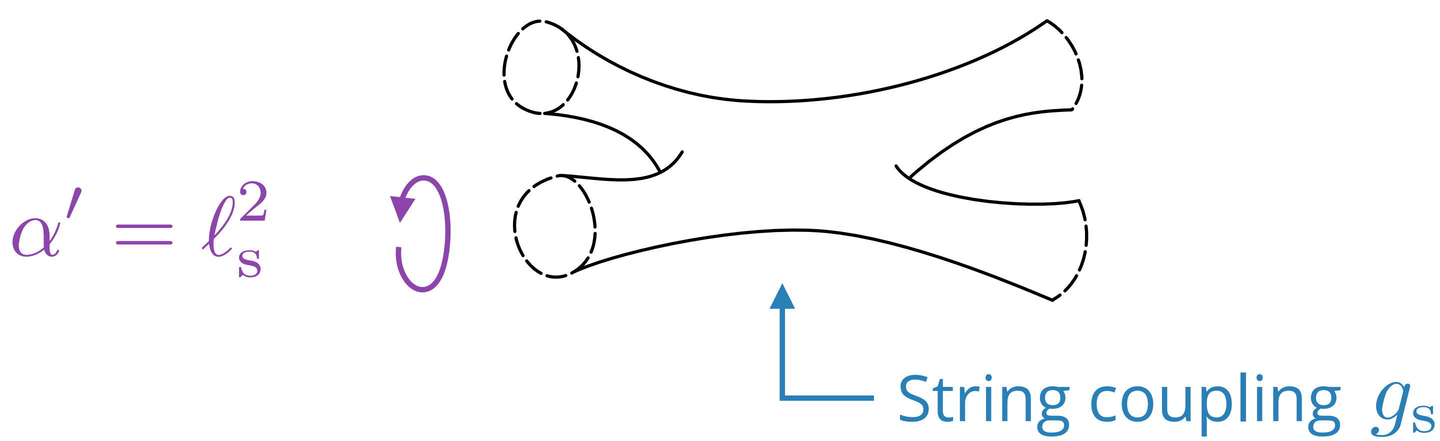

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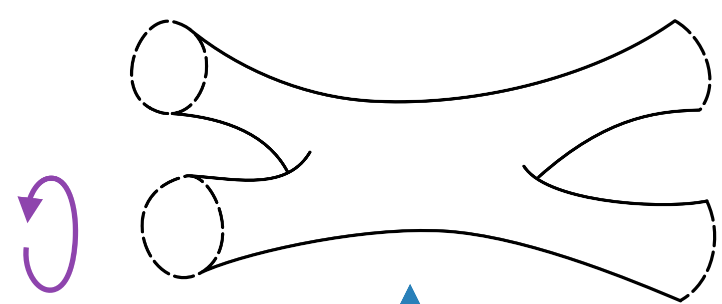
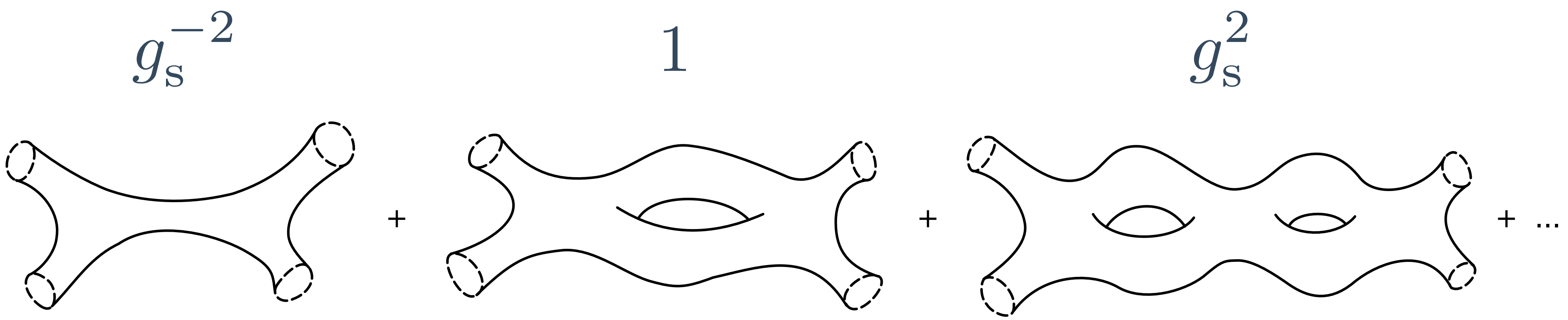


String coupling g_{s}

$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2 \qquad t = -\frac{\alpha'}{4}(k_1 + k_3)^2 \qquad u = -\frac{\alpha'}{4}(k_1 + k_4)^2$$

Interactions

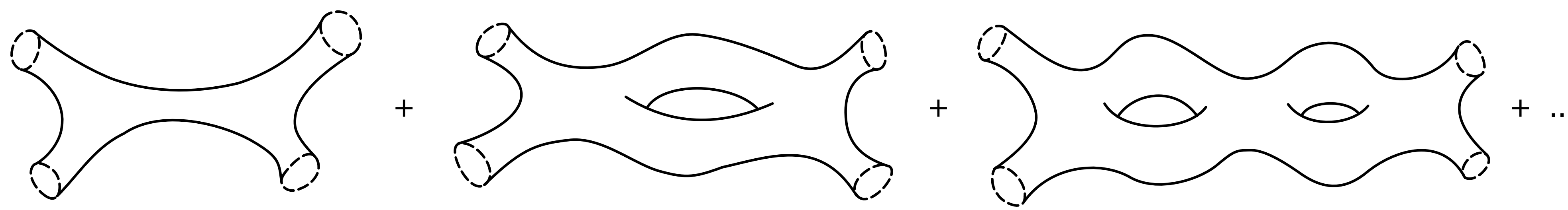


4-graviton amplitude in 10 dimensions:

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions

g_{s}^{-2} $\qquad\qquad 1 \qquad\qquad$ g_{s}^{2}



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$$\mathcal{A} = \left(g_{\mathrm{s}}^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \right) \mathcal{R}^4$$

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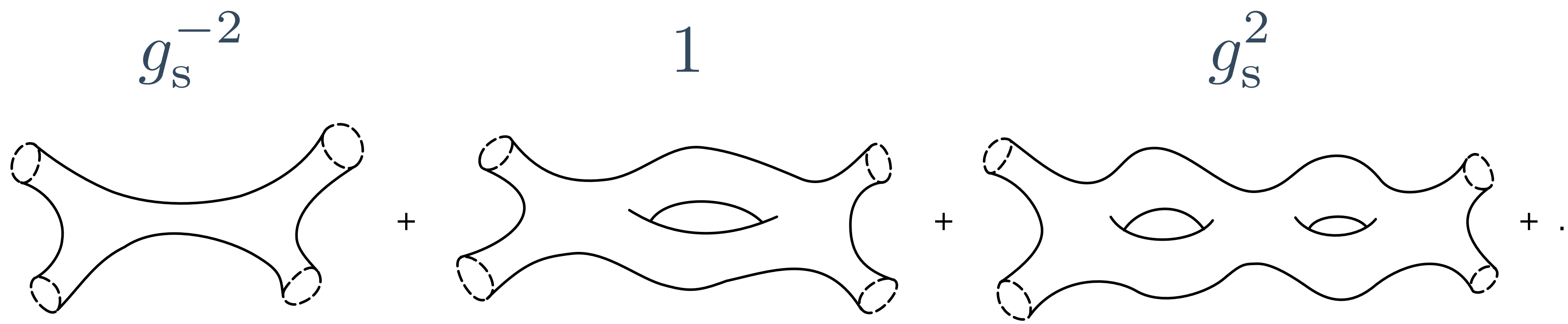

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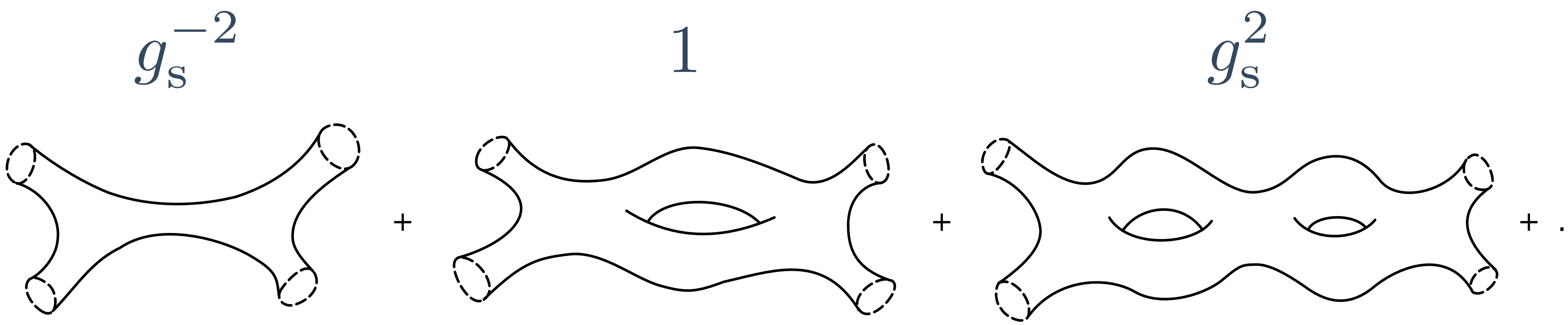
Contraction of 4 linearized Riemann tensors and standard rank 8 tensors $t_8 t_8 \mathcal{R}^4$

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions



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$$\mathcal{A} = \left(g_s^{-2} \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} + 2\pi \int_{\mathscr{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} \mathcal{B}_1(s,t,u;\tau) \right) \mathcal{R}^4$$

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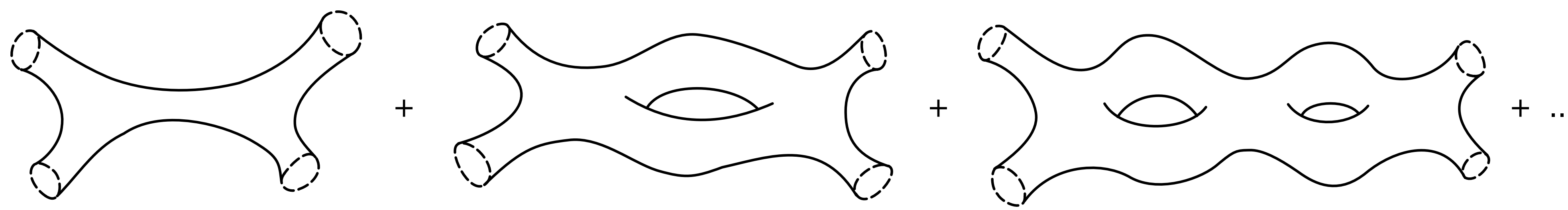
Interactions

g_s^{-2} $\qquad$ 1 $\qquad$ g_s^2



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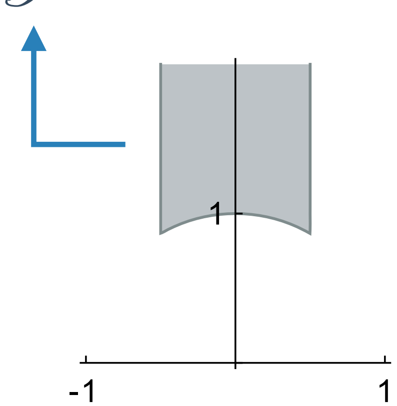

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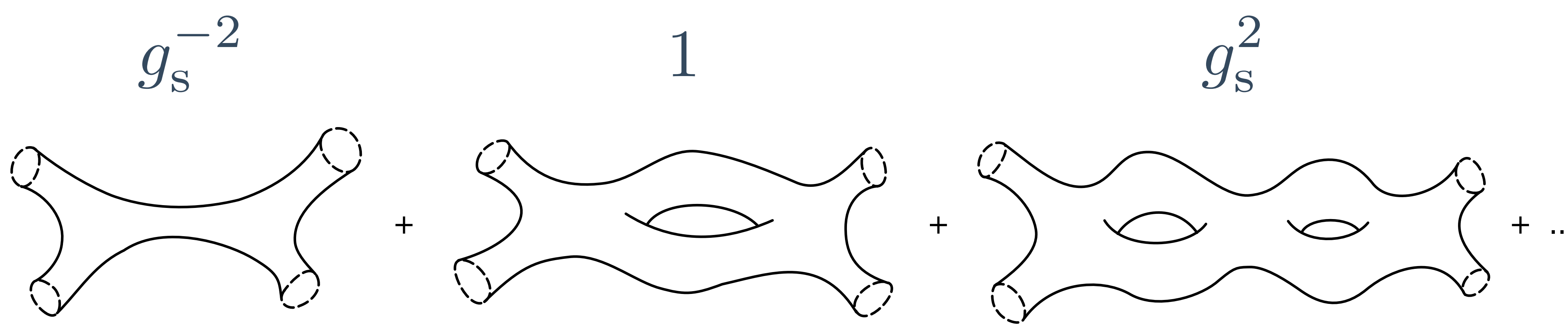

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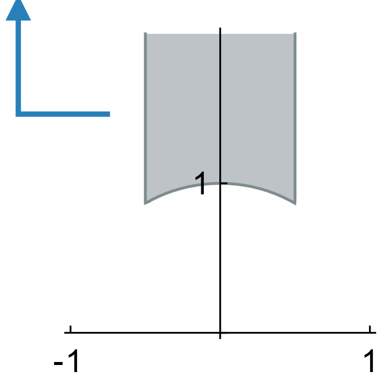
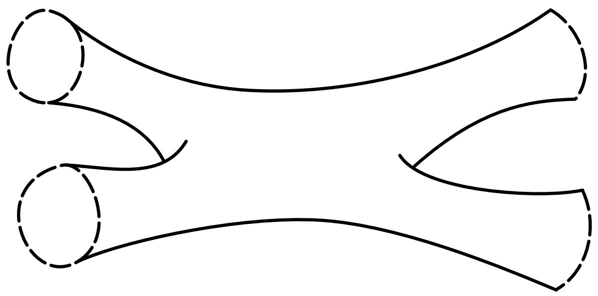


Modular invariant function

[Green-Schwarz, Green-Schwarz-Brink, Gross-Witten]

Interactions

String theory



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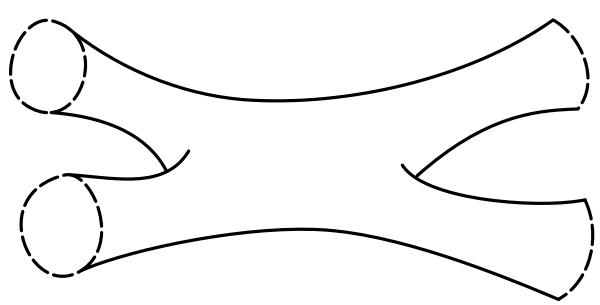
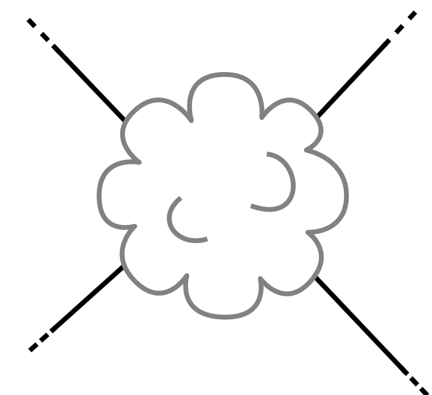


Effective field theory

Supergravity + $\mathcal{O}(\alpha')$



Interactions

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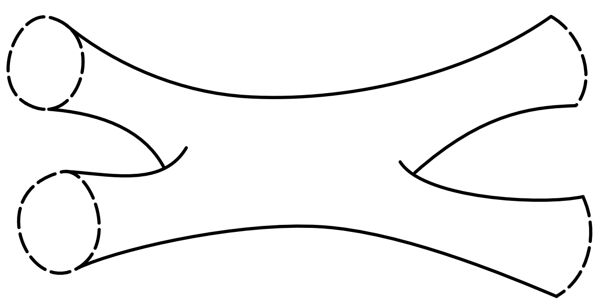

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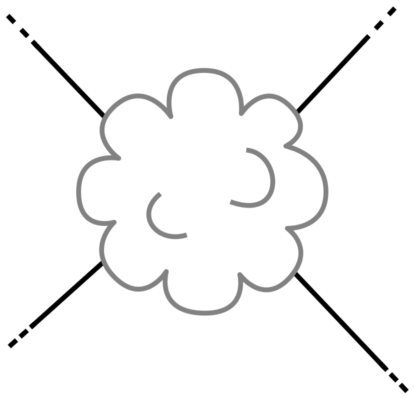
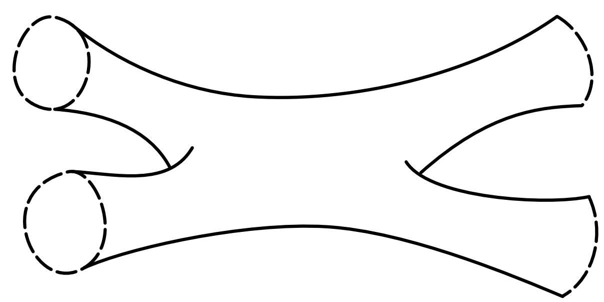


$s, t, u = \mathcal{O}(\alpha')$ $\quad p \longleftrightarrow \partial \implies \alpha'$ -expansion = ∂-expansion

Interactions

String theory



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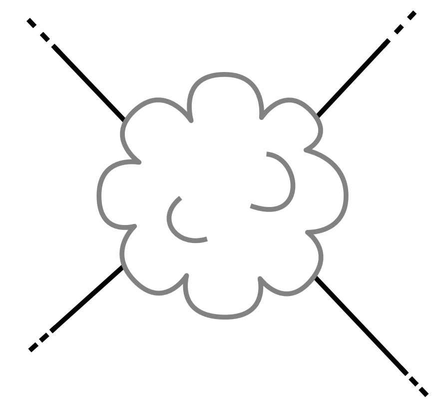
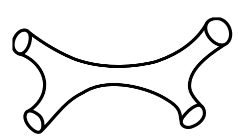
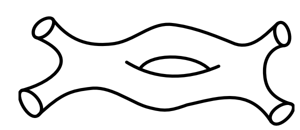


$s, t, u = \mathcal{O}(\alpha')$ $\quad p \longleftrightarrow \partial \implies$ α'-expansion = ∂-expansion





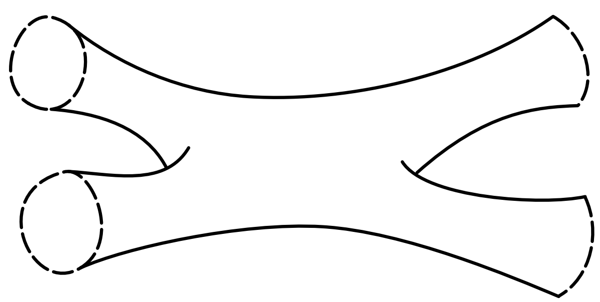
(Einstein frame)

$$\mathcal{L} \propto R + (\alpha')^3 \left(2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2} + \dots \right) R^4 + \dots$$

Interactions

String theory

Supergravity + $\mathcal{O}(\alpha')$



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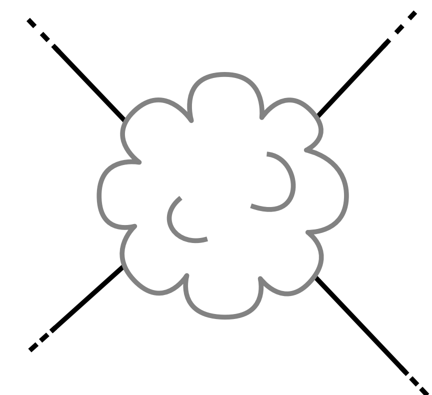

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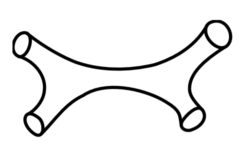
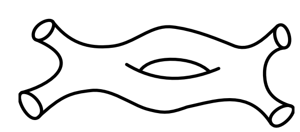




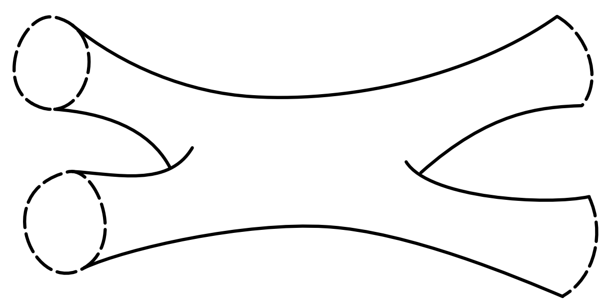
(Einstein frame)

$$\mathcal{L} \propto R + (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \ldots \Big) R^4 + \ldots$$

Contraction of 4 Riemann tensors

Interactions

String theory



Effective field theory $\longrightarrow$

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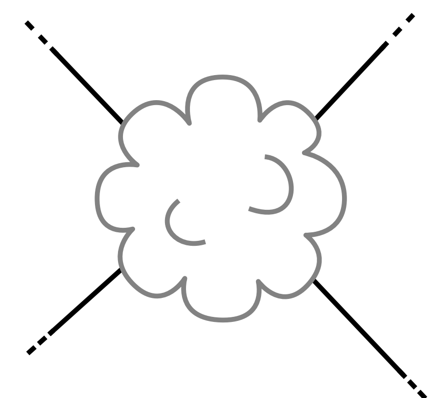

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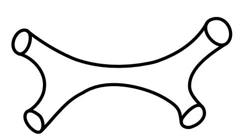

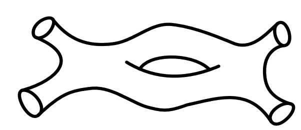

(Einstein frame)

$$
\begin{aligned}
\mathcal{L} \propto R + \; & (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} \; + \; 4\zeta(2) g_s^{1/2} \; + \; \ldots \Big) \quad R^4 + \\
& (\alpha')^5 \Big(\zeta(5) g_s^{-5/2} \; + \; \ldots \Big) D^4 R^4 + \\
& (\alpha')^6 \Big(\tfrac{2}{3}\zeta(3)^2 g_s^{-3} \; + \; \tfrac{4}{3}\zeta(2)\zeta(3) g_s^{-1} \; + \; \ldots \Big) D^6 R^4 + \mathcal{O}((\alpha')^7)
\end{aligned}
$$

Interactions

$$\begin{aligned}\mathcal{L} \propto R + \; & (\alpha')^3\Big(2\zeta(3)g_\mathrm{s}^{-3/2} \; + \; 4\zeta(2)g_\mathrm{s}^{1/2} \qquad\qquad + \; \ldots\Big) \quad R^4 + \\ & (\alpha')^5\Big(\zeta(5)g_\mathrm{s}^{-5/2} \qquad\qquad\qquad\qquad\qquad + \; \ldots\Big)D^4R^4 + \\ & (\alpha')^6\Big(\tfrac{2}{3}\zeta(3)^2g_\mathrm{s}^{-3} \; + \; \tfrac{4}{3}\zeta(2)\zeta(3)g_\mathrm{s}^{-1} \; + \; \ldots\Big)D^6R^4 + \mathcal{O}((\alpha')^7)\end{aligned}$$

Interactions

$$\mathcal{L} \propto R + (\alpha')^3 \Big(2\zeta(3) g_s^{-3/2} + 4\zeta(2) g_s^{1/2} + \dots \Big) R^4 +$$
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$$\mathcal{L} \propto R + (\alpha')^3 \mathcal{E}_0(\tau) R^4 + (\alpha')^5 \mathcal{E}_4(\tau) D^4 R^4 + (\alpha')^6 \mathcal{E}_6(\tau) D^6 R^4 + \dots$$

$$\tau = \chi + i g_s^{-1}$$

Interactions

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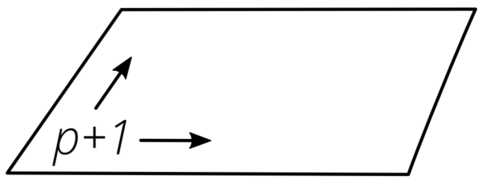
Ahlén, Bao, Basu, Bossard, Cederwall, Fleig, Green, Gubay, Gutperle, HG, Kazhdan, Kiritsis, Kleinschmidt, Lambert, Miller, Nilsson, Obers, Persson, Pioline, Russo, Sethi, Vanhove, Verschinin, Waldron, West, ...

Non-perturbative effects

[Green, Polchinski]

Non-perturbative effects

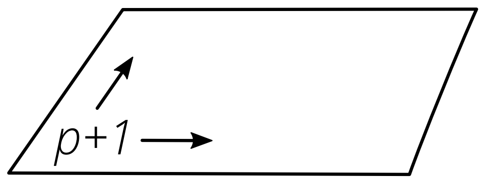


D*p*-brane

p space directions
1 time direction

[Green, Polchinski]

Non-perturbative effects



Dp-brane

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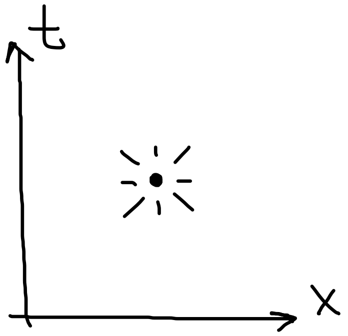
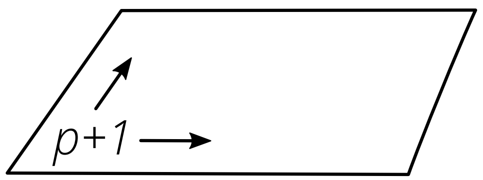


D-instanton

$p = -1$

[Green, Polchinski]

Non-perturbative effects



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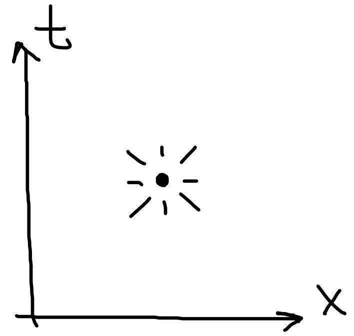
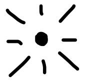


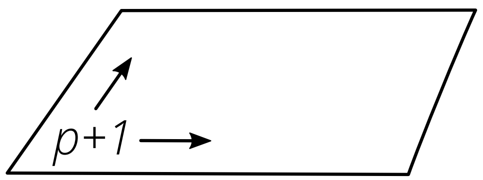
D-instanton

$p = -1$



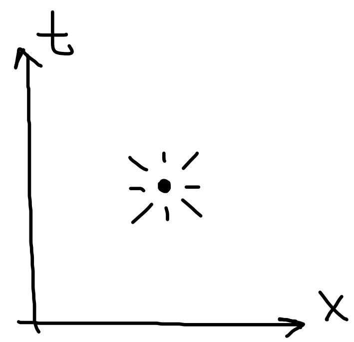
[Green, Polchinski]

Non-perturbative effects



Dp-brane

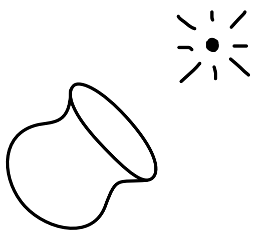
p space directions
1 time direction



D-instanton

$p = -1$

$$\frac{1}{g_s}$$



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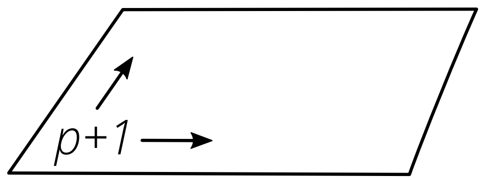

D*p*-brane

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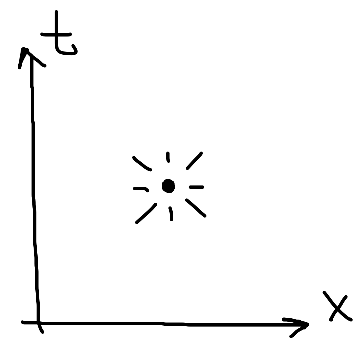

D-instanton

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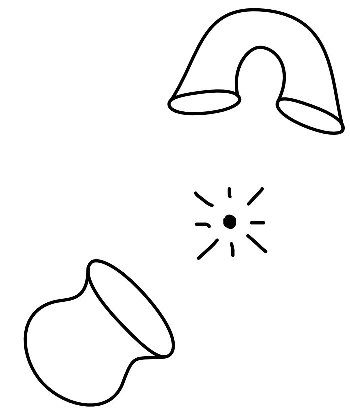

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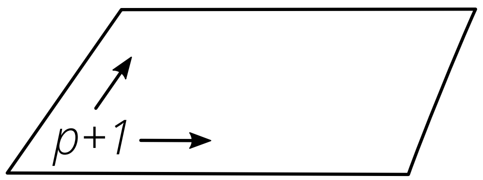
Non-perturbative effects



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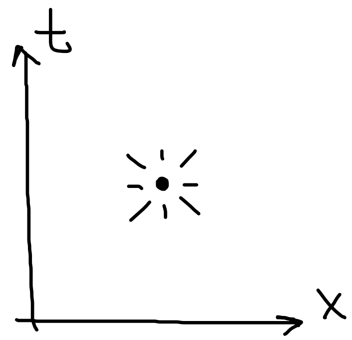

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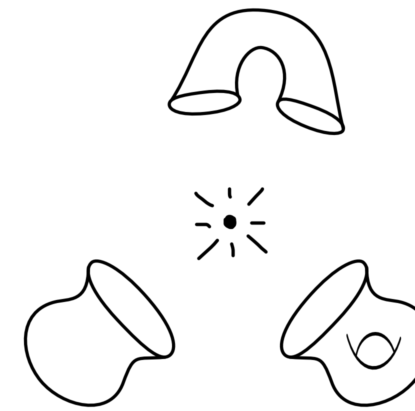

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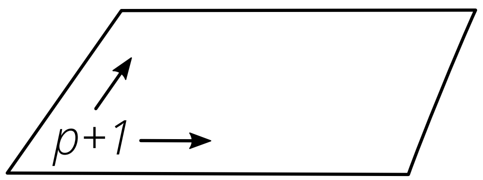

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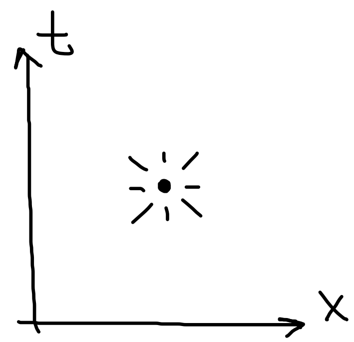

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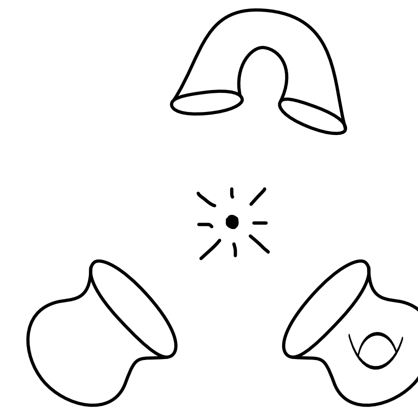

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g_s

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[Green, Polchinski]

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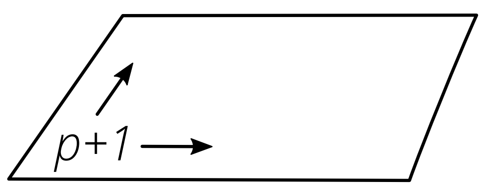

Dp-brane

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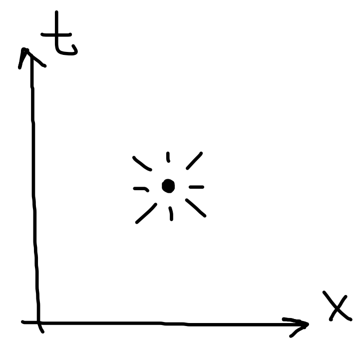

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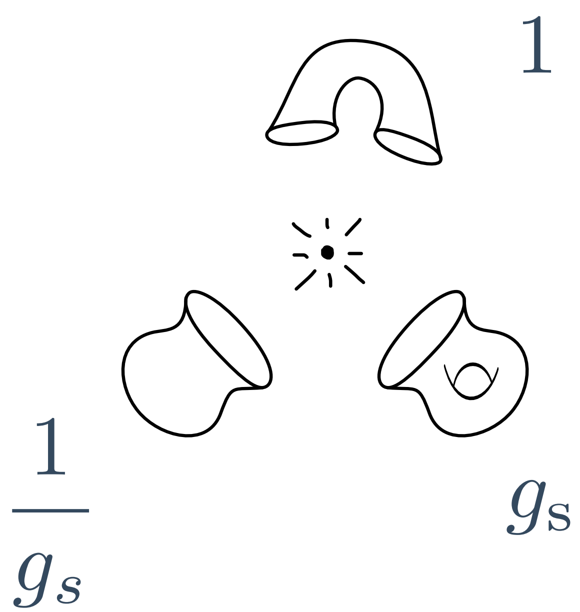
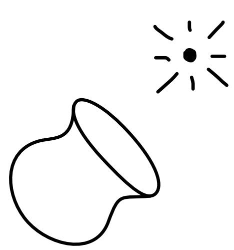


$1 \quad +$



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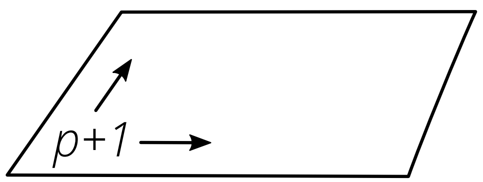

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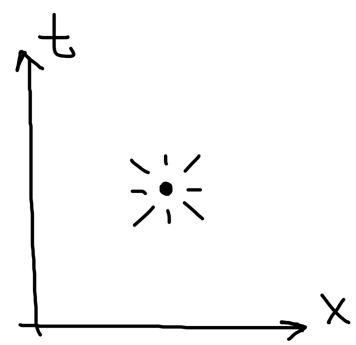

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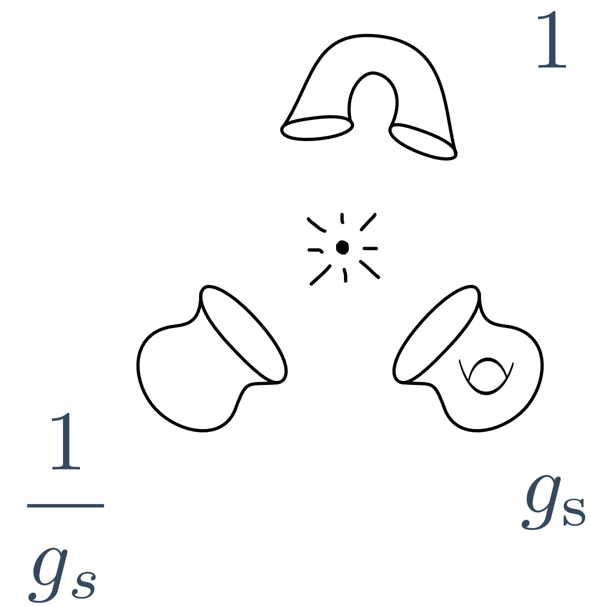
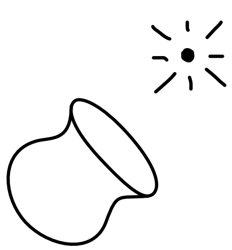
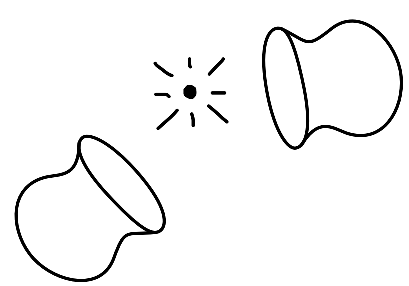


$1 \quad +$  $\quad + \quad \frac{1}{2!}$ 

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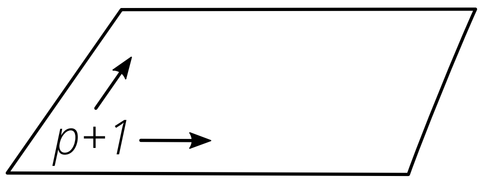

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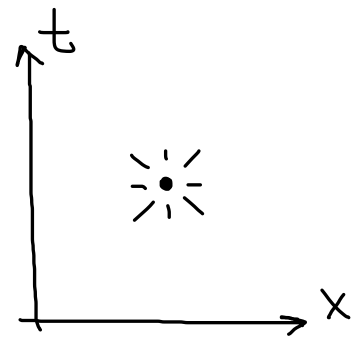

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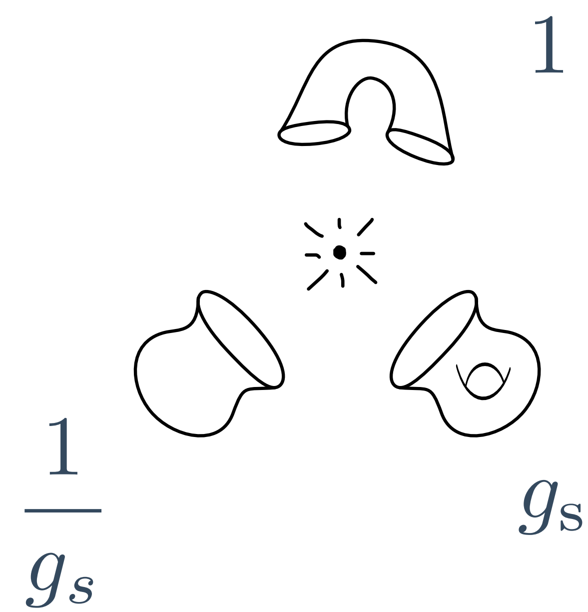
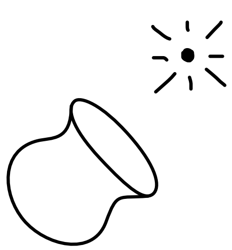
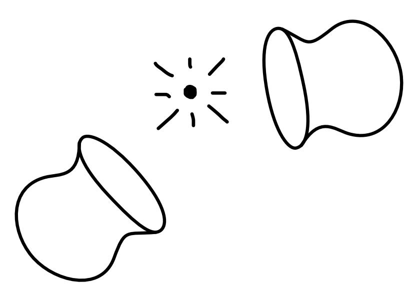


$1 \quad +$  $+ \quad \frac{1}{2!}$ 

Symmetry factor for identical disks

[Green, Polchinski]

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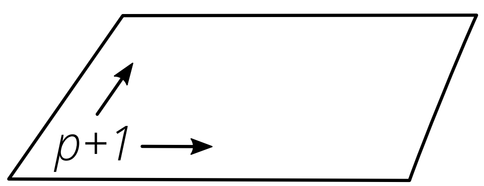

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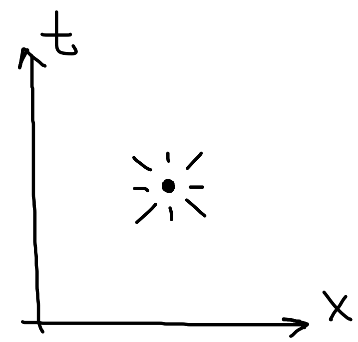

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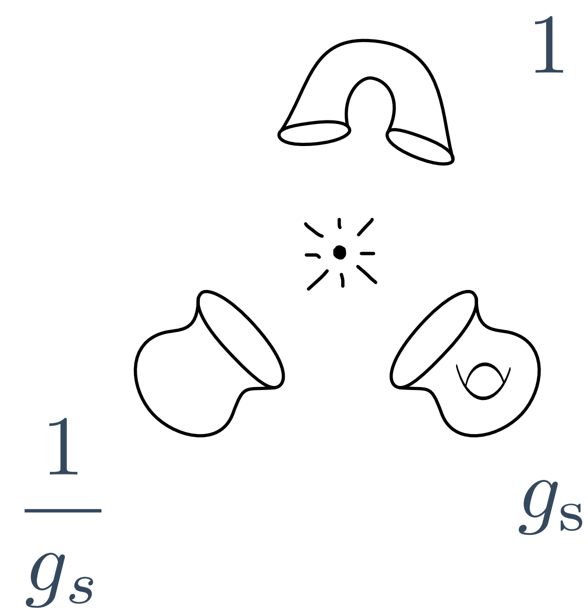

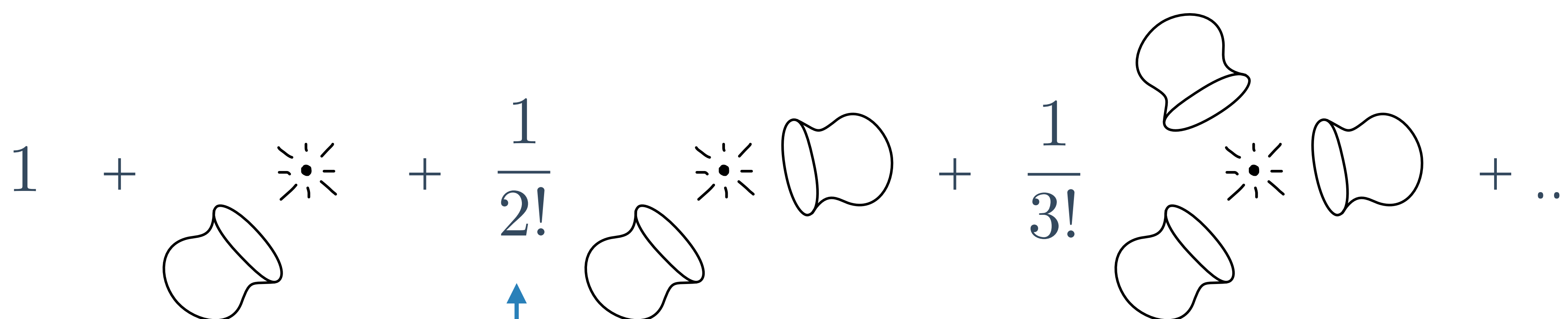

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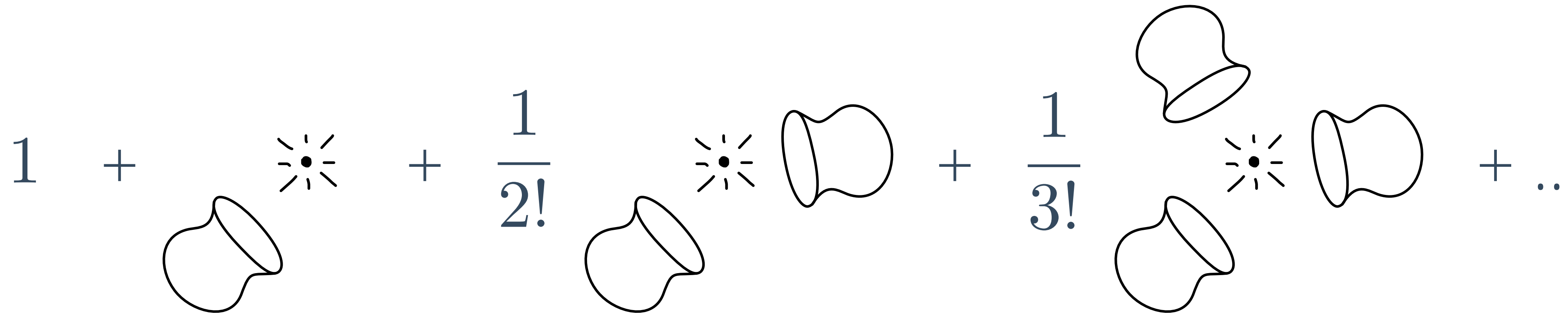

[Green-Gutperle]

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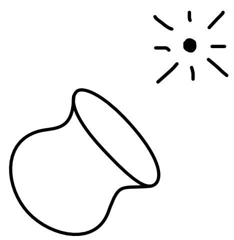
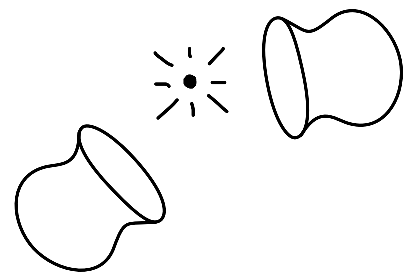
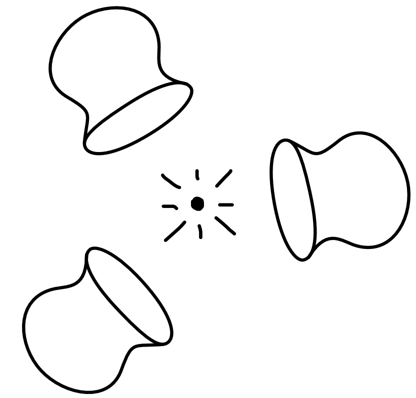
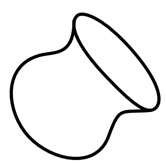
$1 \quad +$  $\quad + \quad \frac{1}{2!}$  $\quad + \quad \frac{1}{3!}$  $\quad + \ \ldots$

$\exp \Bigg($  $\Bigg)$

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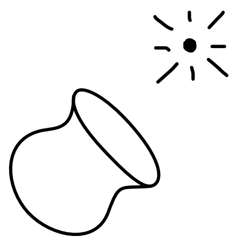
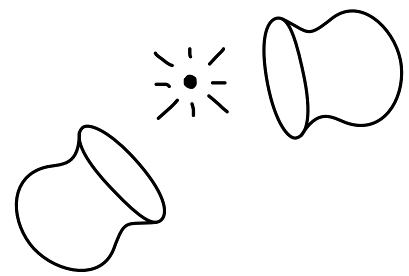
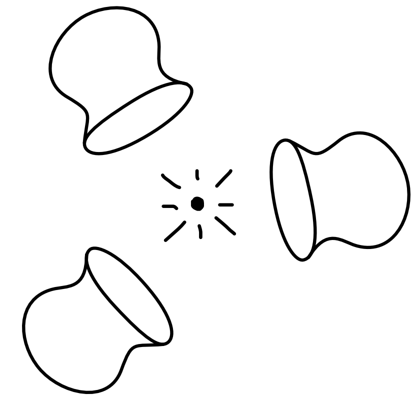
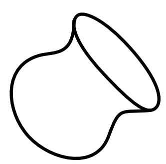
Non-perturbative effects

$$1 +$$  $$+ \frac{1}{2!}$$  $$+ \frac{1}{3!}$$  $$+ \ldots$$

$$\exp\Bigg($$  $$\Bigg) \sim \exp\left(-\frac{\text{const}}{g_{\text{s}}}\right)$$

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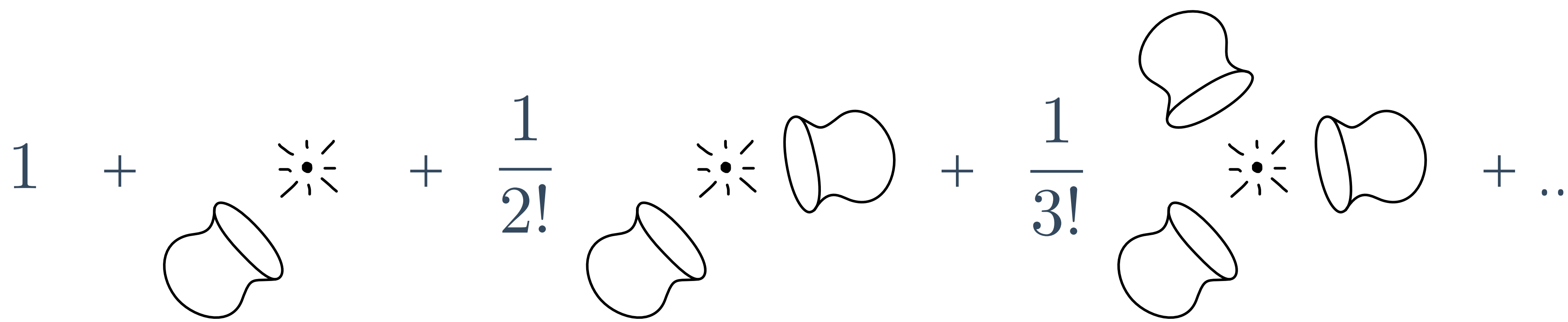
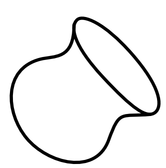


$\exp\Big($  $\Big) \sim \exp\left(-\frac{\mathrm{const}}{g_\mathrm{s}}\right)$

Non-perturbative in g_s

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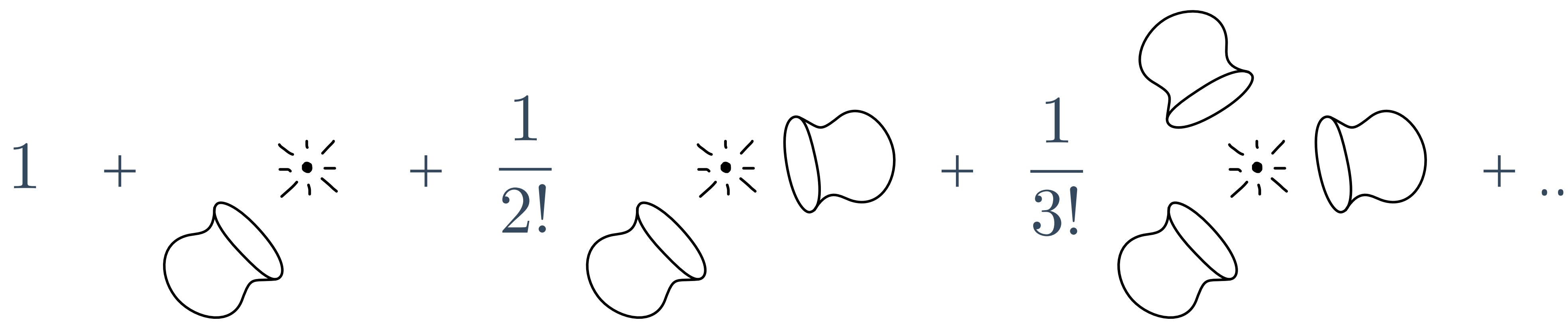
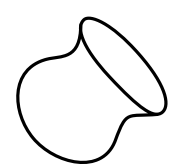


$\exp\Big($  $\Big) \sim \exp\left(-\frac{\text{const}}{g_\text{s}}\right)$

Non-perturbative in g_s

$$\mathcal{E}_0(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + \ldots + Ce^{2\pi i\tau} + \ldots$$

$$\tau = \tau_1 + i\tau_2 = \chi + i g_\text{s}^{-1}$$

[Green-Gutperle]

Moduli space

$$R + (\alpha')^3 \mathcal{E}_0^{(D)}(g) R^4 + (\alpha')^5 \mathcal{E}_4^{(D)}(g) D^4 R^4 + (\alpha')^6 \mathcal{E}_6^{(D)}(g) D^6 R^4 + \ldots$$

$$\mathcal{M}_{\text{classical}} = G(\mathbb{R})/K$$

[Cremmer-Julia]

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D	$G(\mathbb{R})$	K
10	$SL(2,\mathbb{R})$	$SO(2)$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$
7	$SL(5,\mathbb{R})$	$SO(5)$
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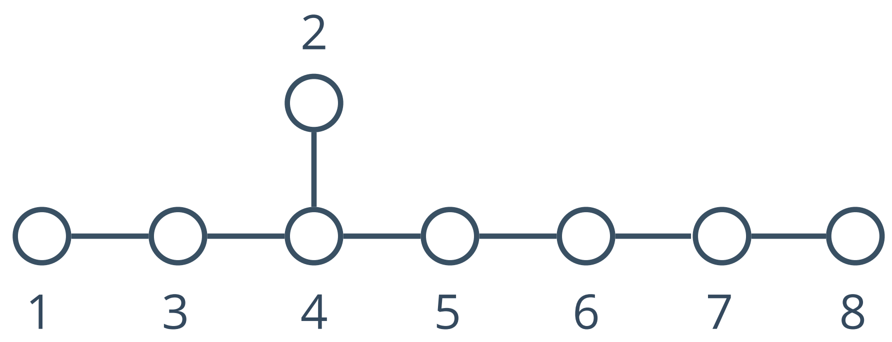

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$$\tau = \chi + i g_{\mathrm{s}}^{-1} \in \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \cong SL(2,\mathbb{R})/SO(2,\mathbb{R})$$

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No similar structure for lower dimensions

U-duality

$G(\mathbb{R}) \circlearrowright \mathcal{M}_{\text{classical}}$ classical symmetry

[Hull-Townsend]

U-duality

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Quantization of charges

[Hull-Townsend]

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Quantization of charges $\Longrightarrow$ classical symmetry $\longrightarrow$ discrete symmetry

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$G(\mathbb{R})$ Chevalley group $G(\mathbb{Z})$

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All observables are invariant under $G(\mathbb{Z})$

[Hull-Townsend]

U-duality

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$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{R}$$

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An *automorphic form* is a smooth function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ satisfying the following conditions

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(B) φ is an eigenfunction under right-translations of $k \in K$

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(D) Growth: for any norm $\|\cdot\|$ on $G(\mathbb{R})$ there exists a positive integer n and constant C such that $|\varphi(g)| \leq C\|g\|^n$

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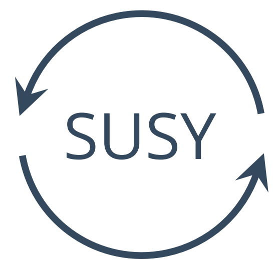
Supersymmetry constraints

bosons	SUSY	fermions
dilaton		dilatino
graviton		gravitino
…		…

[Green-Sethi]

Supersymmetry constraints

bosons



fermions

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10 dimensions:

[Green-Sethi]

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$$\mathcal{L}^{(3)} = \qquad \mathcal{E}_0(\tau) R^4$$

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$$\mathcal{L}^{(3)} = f_{12}(\tau)\lambda^{16} + f_{11}(\tau)\hat{G}\lambda^{14} + \ldots + f_0(\tau)R^4 + \ldots + f_{-12}(\tau)\lambda^{*16}$$

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Linearized SUSY: $$f_{w+1}(\tau) = i\left(\tau_2 \frac{\partial}{\partial \tau} - i\frac{w}{2}\right) f_w(\tau)$$

[Green-Sethi]

Supersymmetry constraints

$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)}$$

$$\delta\Psi = \delta^{(0)}\Psi$$

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Supersymmetry constraints

$$\int d^D x \sqrt{-g}\mathcal{L} = S = S^{(0)} + (\alpha')^3 S^{(3)} + (\alpha')^5 S^{(5)} + \dots$$

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Laplacian on Poincaré UHP

[Green-Sethi]

Supersymmetry constraints

bosons SUSY fermions

10 dimensions: $\Delta = 4\tau_2^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \overline{\tau}}$ Laplacian on Poincaré UHP

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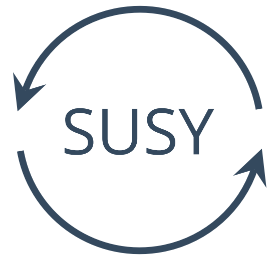
bosons  fermions

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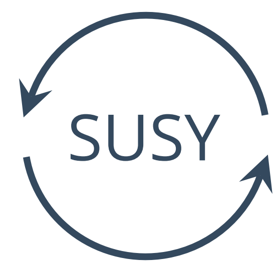
bosons 

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Similarly for lower dimensions

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$E(s;\tau) =$ $\qquad$ $s \in \mathbb{C}$

Eisenstein series

$$E(s;\tau) = \sum_{\substack{c,d\in\mathbb{Z}\\(c,d)\neq(0,0)}} \frac{\mathrm{Im}(\tau)^s}{|c\tau+d|^{2s}}$$

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Multiplicative character trivially extended to $G(\mathbb{R})$

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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

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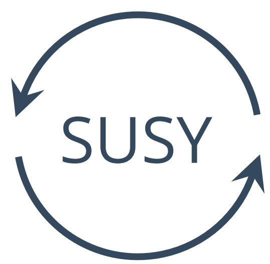

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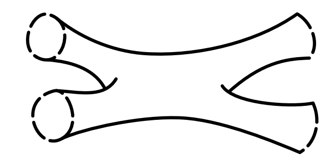

(Einstein frame)

$$(\Delta - \tfrac{3}{4})\mathcal{E}_0(\tau) = 0$$

$$(\Delta - \tfrac{15}{4})\mathcal{E}_4(\tau) = 0$$

$$\mathcal{E}_0(\tau) \sim 2\zeta(3)\tau_2^{3/2}$$

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[Green-Gutperle, Pioline, Green-Russo-Vanhove]

Eisenstein series

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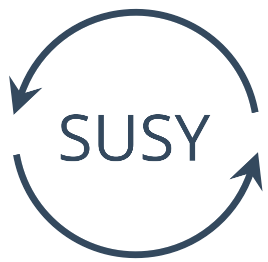

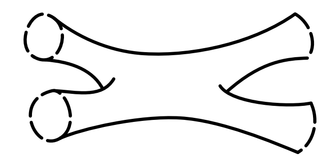

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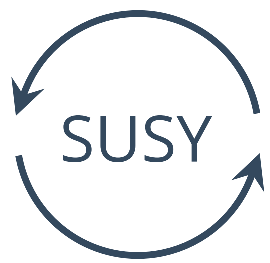
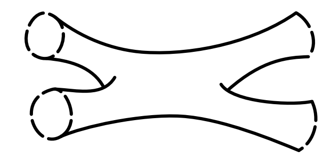
[Green-Gutperle, Pioline, Green-Russo-Vanhove]

Eisenstein series





$$\mathcal{E}_0(\tau) = 2\zeta(3)E(3/2;\tau)$$

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$\mathcal{E}_6(\tau)$ as a sum over images $\sum_{B(\mathbb{Z})\backslash G(\mathbb{Z})}$ but not of a character χ

[Green-Miller-Vanhove]

Extracting physical information

Expand Bessel function in g_{s} $\tau = \chi + i g_{\mathrm{s}}^{-1}$

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$$\mathcal{E}_0(\tau) = 2\zeta(3) g_{\mathrm{s}}^{-3/2} + 4\zeta(2) g_{\mathrm{s}}^{1/2} + 2\pi \sum_{m\neq 0} \sqrt{|m|}\sigma_{-2}(m) e^{-2\pi|m|g_{\mathrm{s}}^{-1} + 2\pi i m\chi}\left[1 + \mathcal{O}(g_{\mathrm{s}})\right]$$

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Perturbative
(zero-mode)

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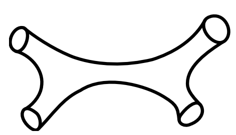 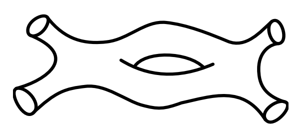

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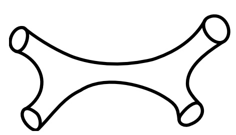

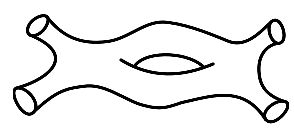

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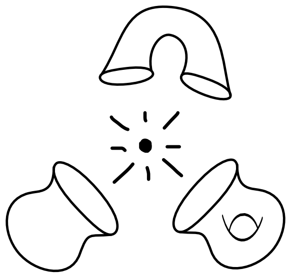

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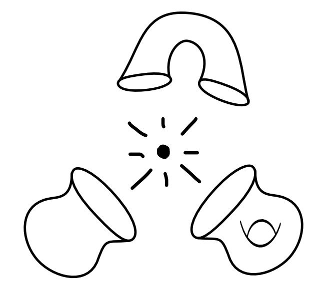

Instanton action

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Sums over the number of ways the charge *m* can be factorised into two integers

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Sums over the number of ways the charge *m* can be factorised into two integers

wrapping number of a T-dual D-particle and charge

[Green-Gutperle]

Lower dimensions

Lower dimensions

D	$G(\mathbb{R})$	K	$G(\mathbb{Z})$
10	$SL(2,\mathbb{R})$	$SO(2)$	$SL(2,\mathbb{Z})$
9	$SL(2,\mathbb{R}) \times \mathbb{R}^+$	$SO(2)$	$SL(2,\mathbb{Z}) \times \mathbb{Z}_2$
8	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$
7	$SL(5,\mathbb{R})$	$SO(5)$	$SL(5,\mathbb{Z})$
6	$Spin(5,5;\mathbb{R})$	$\big(Spin(5) \times Spin(5)\big)/\mathbb{Z}_2$	$Spin(5,5;\mathbb{Z})$
5	$E_6(\mathbb{R})$	$USp(8)/\mathbb{Z}_2$	$E_6(\mathbb{Z})$
4	$E_7(\mathbb{R})$	$SU(8)/\mathbb{Z}_2$	$E_7(\mathbb{Z})$
3	$E_8(\mathbb{R})$	$Spin(16)/\mathbb{Z}_2$	$E_8(\mathbb{Z})$

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$$E(\chi;g)=\sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})}\chi(\gamma g)$$

Parabolic subgroups

Fourier expand
in different directions $\longleftrightarrow$ Unipotent subgroup U

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$\uparrow$

Choice of parabolic subgroup P

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Choice of parabolic subgroup P

Σ choice of simple roots $\langle \Sigma \rangle$ generated root system

Parabolic subgroups

Fourier expand
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Choice of parabolic subgroup P

Σ choice of simple roots

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$G = SL(4)$

$\Sigma = \{\alpha_1\}$

Parabolic subgroups

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$P = LU$

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Choice of parabolic subgroup P

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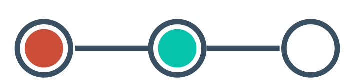

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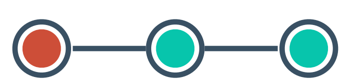
$G = SL(4)$  $\Sigma = \{\alpha_1\}$

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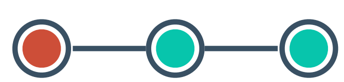

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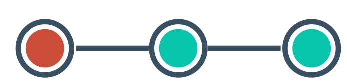

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Parabolic subgroups

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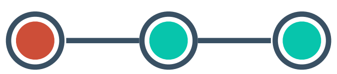

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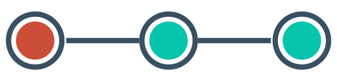

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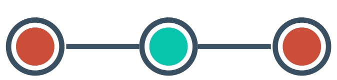

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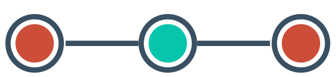

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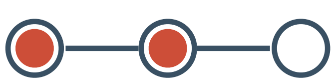

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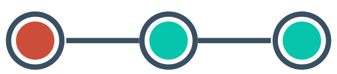

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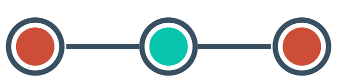

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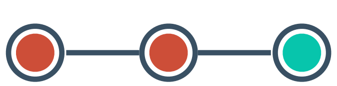

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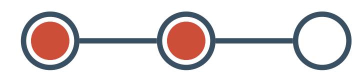
Parabolic subgroups

Parabolic subgroups

Maximal parabolic

Parabolic subgroups

Minimal parabolic
Borel



Maximal parabolic

Parabolic subgroups

Minimal parabolic
Borel

$$B = NA$$

$$N = \left\{ \begin{pmatrix} \boxed{1} & * & * & * \\ & \boxed{1} & * & * \\ & & \boxed{1} & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Maximal parabolic

$$P = LU$$

$$U = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & \boxed{1} \end{pmatrix} \right\}$$

Fourier expansion

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Let $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \underset{\cong\, S^1}{U(1)}$ be a multiplicative character

Parametrised by $m_\alpha \in \mathbb{Z}$ called charges

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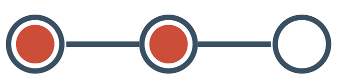

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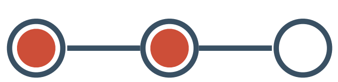

$$\psi_U\left(\begin{pmatrix} 1 & & & y_1 \\ & 1 & & y_2 \\ & & 1 & y_3 \\ & & & 1 \end{pmatrix}\right) = e^{2\pi i(m_1 y_1 + m_2 y_2 + m_3 y_3)}$$

Fourier expansion

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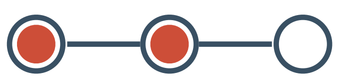

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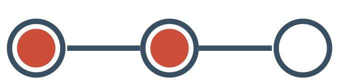

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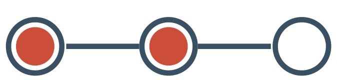

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Fourier expansion

Fourier expansion

$$E(\chi; g) = \sum_{\psi} F_U(\chi, \psi; g)$$

Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi \neq 1} F_U(\chi, \psi; g)$$

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Fourier expansion

$$E(\chi; g) = F_U(\chi, 1; g) + \sum_{\psi^{(1)} \neq 1} F_{U^{(1)}}(\chi, \psi^{(1)}; g) + \sum_{\psi^{(2)} \neq 1} F_{U^{(2)}}(\chi, \psi^{(2)}; g) + \ldots$$

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$$F_U(\chi,\psi;ug) = \psi(u)F_U(\chi,\psi;g) \qquad \psi(u_1u_2) = \psi(u_1)\psi(u_2)$$

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Terminology

$P = B \longrightarrow U = N$ Fourier coefficient is a Whittaker coefficient

$$N = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}$$

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Characters and coefficients with all $m_\alpha \neq 0$ are called generic otherwise they are called degenerate

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

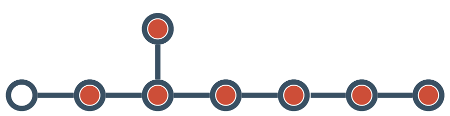
[Green-Miller-Vanhove]

Fourier expansion

Choice of unipotent subgroup U $\longleftrightarrow$ Study different perturbative and non-perturbative effects

- String perturbation limit
  D-instantons | NS5-instantons

$g_{\mathrm{s}} \to 0$



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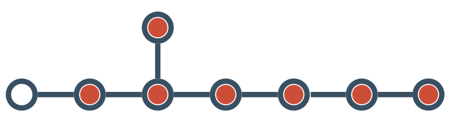
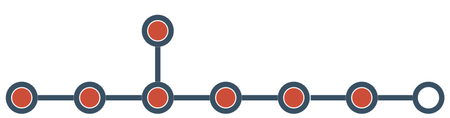


- **Decompactification limit**
  Higher dimensional black holes | BPS states

  Large radius for compactified circle



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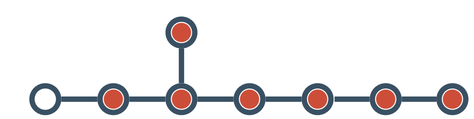

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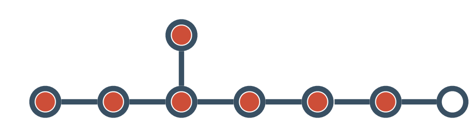

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  M2, M5-instantons
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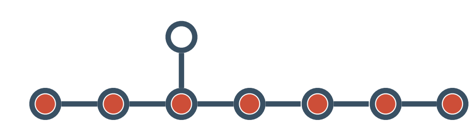

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  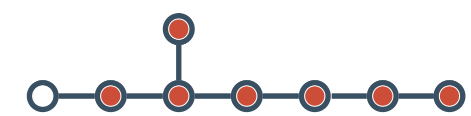

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  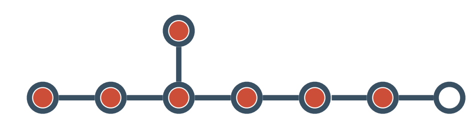

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  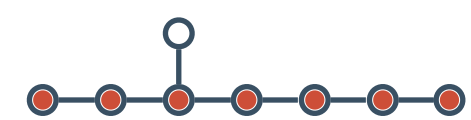

[Green-Miller-Vanhove]

Maximal parabolic subgroups

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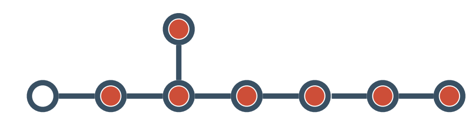

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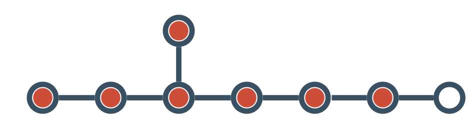

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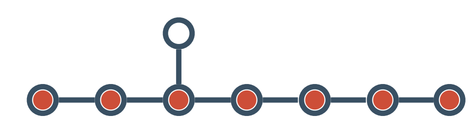

Maximal parabolic subgroups

[Green-Miller-Vanhove]

Difficult to compute!

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  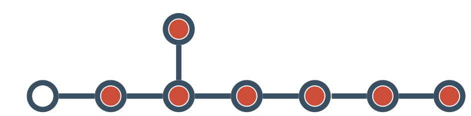

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  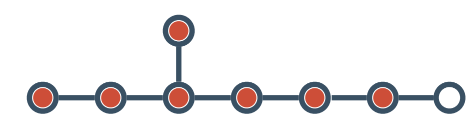

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  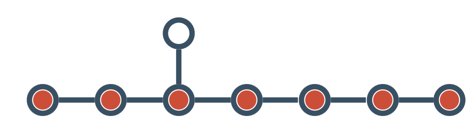

[Green-Miller-Vanhove]

Maximal parabolic subgroups

Difficult to compute!

Recent result in [Bossard-Pioline]

Fourier expansion

Goal: find expressions for Fourier coefficients
in terms of (known) Whittaker coefficients

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Would allow us to compute non-perturbative effects that capture information about instantons and black holes

Adelic framework

> *An efficient, but abstract, way to approach the subject of automorphic forms is by the introduction of adeles, rather ungainly objects that nevertheless, once familiar, spare much unnecessary thought and many useless calculations.*

— Robert P. Langlands*

*Representation theory - its rise and its role in number theory, Proceedings of the Gibbs symposium (1989)

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Eisenstein series

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$$\begin{array}{c}\text{Adelic Eisenstein series}\\ \Big\uparrow \text{Lift}\\ \text{Eisenstein series}\end{array}$$

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$$
\begin{tikzcd}
\text{Adelic Eisenstein series} \arrow[r, "\text{Compute}"] & \text{Adelic Fourier coefficient} \arrow[d, "\text{Restrict}"] \\
\text{Eisenstein series} \arrow[u, "\text{Lift}"'] & \text{Fourier coefficient}
\end{tikzcd}
$$

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The adeles

For a prime p

$\mathbb{Q}$

The adeles

For a prime p

$$\mathbb{Q} \xrightarrow{\text{Euclidean norm}} \mathbb{R}$$

The adeles

For a prime p

$$
\begin{tikzcd}
 & \mathbb{R} \\
\mathbb{Q} \arrow[ur, "\text{Euclidean norm}"] \arrow[dr, "p\text{-adic norm}"'] & \\
 & \mathbb{Q}_p
\end{tikzcd}
$$

p-adic numbers

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p-adic numbers

$$|p_1^{k_1} \cdots p_r^{k_r}|_{p_i} = p_i^{-k_i}$$

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The adeles are then defined as

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p \qquad x = (x_\infty; x_2, x_3, x_5, \ldots) \in \mathbb{A}$$

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$\mathbb{Q}_p$ *p*-adic numbers

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$$\mathbb{Q} \hookrightarrow \mathbb{A}$$
$$q \mapsto (q; q, q, \ldots)$$

$\mathbb{Q}$ is discrete in $\mathbb{A}$ taking the role of $\mathbb{Z}$ in $\mathbb{R}$

Much easier to work with since it is a field!

Adelic framework

$$\mathcal{E}_0^{(D)}(g), \quad \mathcal{E}_4^{(D)}(g), \quad \mathcal{E}_6^{(D)}(g) \qquad : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$$

Adelic framework

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Lift to the adeles [FGKP15 §4.2.2]

$$G(\mathbb{A}) = G(\mathbb{R}) \times \prod_{p \text{ prime}}{}' G(\mathbb{Q}_p) \qquad K_{\mathbb{A}} = K \times \prod_{p \text{ prime}} G(\mathbb{Z}_p)$$

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Adelic framework

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Eisenstein series $\longrightarrow$ Adelic Eisenstein series

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$$\sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \chi_{\mathbb{R}}(\gamma g) \qquad\qquad \sum_{\gamma \in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_{\mathbb{A}}(\gamma g)$$

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Fourier coefficients $\longrightarrow$ Adelic Fourier coefficients

Adelic framework

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$$\int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(\chi; ug)\overline{\psi_{\mathbb{R}}(u)}\, du \qquad \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} E(\chi; ug)\overline{\psi_{\mathbb{A}}(u)}\, du$$

$$m_\alpha \in \mathbb{Z} \qquad m_\alpha \in \mathbb{Q}$$

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Computing adelic Fourier coefficients

[FGKP15 §9-10]

Whittaker coefficients

Constant term: Langlands' constant term formula

Computing adelic Fourier coefficients

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Computing adelic Fourier coefficients

[FGKP15 §9-10]

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Maximally degenerate

Computing adelic Fourier coefficients

[FGKP15 §9-10]

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Factorisation

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Maximally degenerate   Factorisation

[GKP14]

Fourier coefficients

In terms of Whittaker coefficients
Simplify drastically for certain χ

Example of simplifications

$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

[FGKP15 §10.6]

Example of simplifications

$G = SL(3)$ $\qquad E(\chi; g)$ $\qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$

$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ $\qquad \psi_{m_1,m_2}\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i (m_1 x_1 + m_2 x_2)}$

[FGKP15 §10.6]

Example of simplifications

$$G = SL(3) \qquad E(\chi; g) \qquad \chi \longleftrightarrow (s_1, s_2) \in \mathbb{C}^2$$

$$N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\} \qquad \psi_{m_1,m_2}\left(\begin{pmatrix} 1 & x_1 & * \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\right) = e^{2\pi i (m_1 x_1 + m_2 x_2)}$$

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

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$$W_N(\chi, \psi_{m_1, m_2}; g) \propto \underbrace{\left(\begin{smallmatrix}\text{arithmetic} \\ \text{factor}\end{smallmatrix}\right)}_{\text{p-adic part}} \int K_{\#}(\ldots)\, K_{\#}(\ldots)$$

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p-adic part

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots)$$

Vanishes for certain (s_1, s_2)

[FGKP15 §10.6]

Example of simplifications

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\ \text{factor}}\right) \int K_\#(\ldots)\, K_\#(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\substack{\text{arithmetic}\\ \text{factor}}\right) \int K_{\#}(\ldots)\, K_{\#}(\ldots)$$

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Example of simplifications

Certain (s_1, s_2)

$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \binom{\text{arithmetic}}{\text{factor}} \int K_\#(\ldots)\, K_\#(\ldots) \longrightarrow 0$$

[FGKP15 §10.6]

Example of simplifications

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_{\#}(\ldots)\, K_{\#}(\ldots) \longrightarrow 0$$

$$W_N(\chi, \psi_{m_1,0}; g) \propto K_{\#}(\ldots) + K_{\#}(\ldots) + K_{\#}(\ldots)$$

[FGKP15 §10.6]

Example of simplifications

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$$W_N(\chi, \psi_{m_1,m_2}; g) \propto \left(\begin{smallmatrix}\text{arithmetic}\\ \text{factor}\end{smallmatrix}\right) \int K_\#(\ldots)\, K_\#(\ldots) \longrightarrow 0$$

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[FGKP15 §10.6]

Example of simplifications

Certain (s_1, s_2)

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$$W_N(\chi, \psi_{m_1,0}; g) \propto K_\#(\dots) + K_\#(\dots) + K_\#(\dots) \longrightarrow K_\#(\dots)$$

To explain this, we need to study
small automorphic representations

[FGKP15 §10.6]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowright$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

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[Bump, Goldfeld-Hundley]

Automorphic representations

$G(\mathbb{A}) \circlearrowleft$ Space of automorphic forms*

Automorphic representation π = an irreducible component of the above space under this action

What is a small automorphic representation?

* With some subtleties described in [FGKP15 §6]

[Bump, Goldfeld-Hundley]

Wavefront set

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

The (global) wavefront set contains all the characters ψ which can give rise to non-vanishing Fourier coefficients in that representation

$$\psi \notin \mathrm{WF}(\pi) \implies F_U(\chi, \psi; g) = 0 \quad \text{for} \quad E(\chi; g) \in \pi$$

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Small automorphic representations have few non-vanishing Fourier coefficients

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters ψ $\longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

Wavefront set

Characters $\psi \longleftrightarrow$ Nilpotent elements in $\mathfrak{g}(\mathbb{Q})$

Nilpotent orbit $\quad \mathcal{O} = \{gXg^{-1} \mid g \in G(\mathbb{C})\} \quad X \in \mathfrak{g}$ nilpotent

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$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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So called admissible orbits

[Mœglin–Waldspurger, Matumoto, Ginzburg-Rallis-Soudry, Ginzburg, Gomez-Gourevitch-Sahi, Jiang-Liu-Savin, Joseph, Miller-Sahi]

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Closure with respect to partial ordering

$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i}$$

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Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

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$$(p_1, p_2, \ldots)$$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$(p_1, p_2, \ldots)$

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$

partial ordering

Nilpotent orbits

[Collingwood-McGovern]

For $SL(n)$, orbits can be identified with partitions of n

decreasing order

$$(p_1, p_2, \ldots) \leq (q_1, q_2, \ldots)$$

partial ordering

$$\Longleftrightarrow$$

$$\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i \quad \forall k$$

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Illustrated by a Hasse diagram

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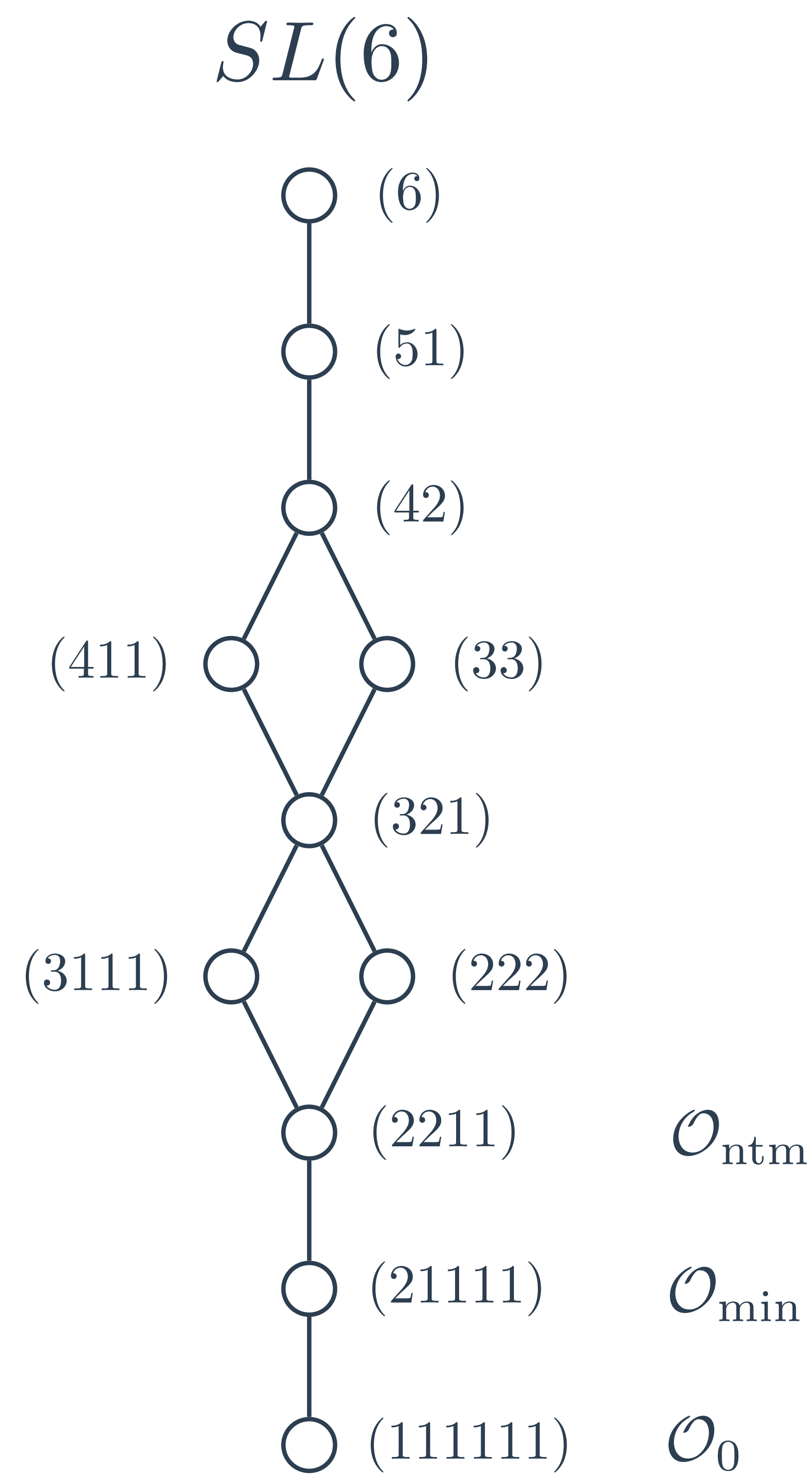

Nilpotent orbits

[Collingwood-McGovern]

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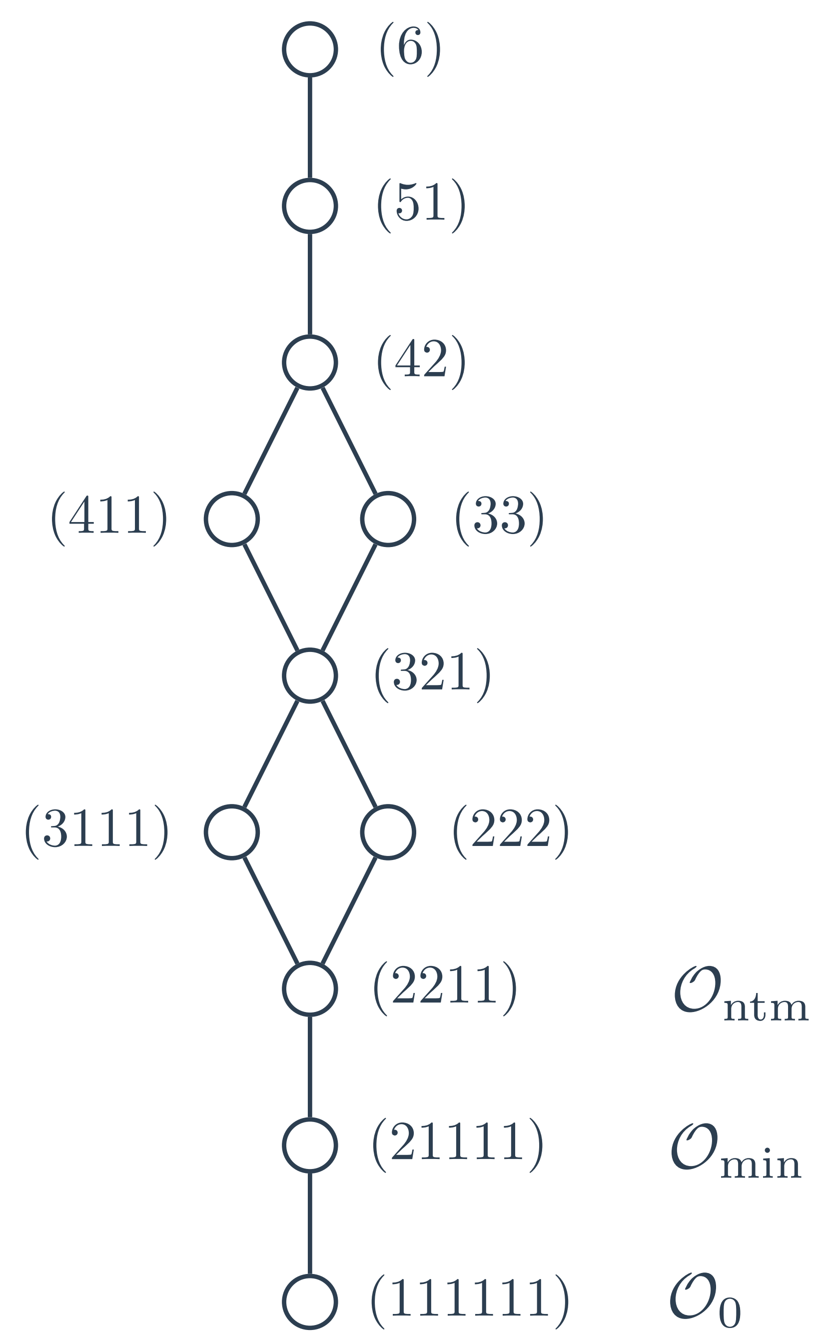
Illustrated by a Hasse diagram

Closure: $\overline{\mathcal{O}} = \bigcup\limits_{\mathcal{O}' \leq \mathcal{O}} \mathcal{O}'$

$SL(6)$



Automorphic representations

Small representations

Automorphic representations

Small representations

$$\mathrm{WF}(\pi_{\mathrm{min}}) = \overline{\mathcal{O}_{\mathrm{min}}} = \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

$$\mathrm{WF}(\pi_{\mathrm{ntm}}) = \overline{\mathcal{O}_{\mathrm{ntm}}} = \mathcal{O}_{\mathrm{ntm}} \cup \mathcal{O}_{\mathrm{min}} \cup \mathcal{O}_0$$

Automorphic representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\qquad$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ $\qquad$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

Small representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

Automorphic representations

Small representations

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$\mathcal{E}_0^{(D)} \in \pi_{\mathrm{min}}$ $\quad$ $\mathcal{E}_4^{(D)} \in \pi_{\mathrm{ntm}}$ $\quad$ [Green-Miller-Vanhove, Pioline, Bossard-Verschinin]

Certain (s_1, s_2) $\longleftrightarrow$ χ_{min} such that $E(\chi_{\mathrm{min}}, g) \in \pi_{\mathrm{min}}$

$\int K\,K \longrightarrow 0$

$\sum K \longrightarrow K$

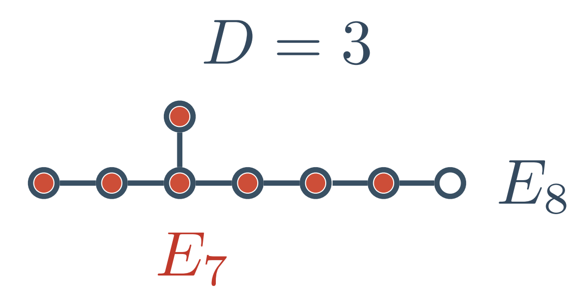
Automorphic representations

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Automorphic representations

- Decompactification limit
  Higher dimensional black holes | BPS states

Large radius for
compactified circle



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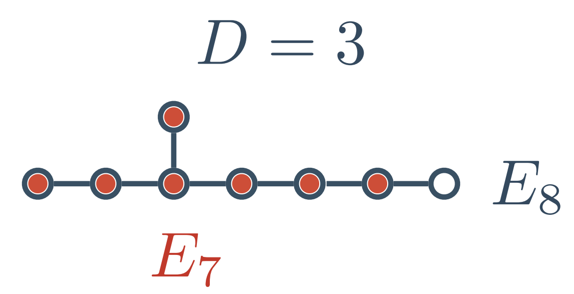

π_{min} $\qquad$ π_{ntm} $\qquad$ π_{3A_1} $\qquad$ π_{A_2}

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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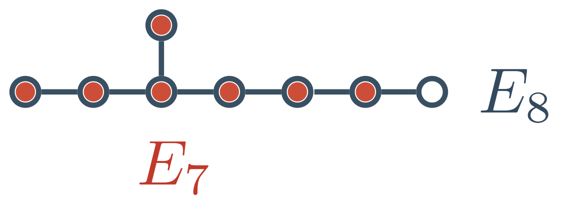
Large radius for compactified circle

$D = 3$



$\mathcal{E}_0^{(D)}$ $\mathcal{E}_4^{(D)}$

π_{min} π_{ntm} π_{3A_1} π_{A_2}

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

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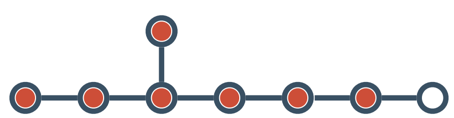

	$\mathcal{E}_0^{(D)}$	$\mathcal{E}_4^{(D)}$			
	π_{min}	π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \text{WF}\}$	28	45	55	56	[Miller-Sahi]

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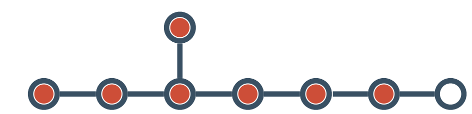

	$\mathcal{E}_0^{(D)}$ $\pi_{\min}$	$\mathcal{E}_4^{(D)}$ π_{ntm}	π_{3A_1}	π_{A_2}	
$\dim\{\psi_U \in \mathrm{WF}\}$	28	45	55	56	[Miller-Sahi]
$D=4$ BPS-orbits $E_7 \circlearrowleft \{\psi_U\}$	½ BPS	¼ BPS	⅛ BPS	⅛ BPS$^+$	

[Ferrara-Günaydin, Ferrara-Maldacena, Green-Miller-Vanhove]

Goal: find expressions for Fourier coefficients in terms of (known) Whittaker coefficients using vanishing properties of the given π

Previous results

[Miller-Sahi]

Previous results

Theorem

For $G = E_6, E_7$, an automorphic form $\varphi \in \pi_{\min}$ is completely determined by maximally degenerate Whittaker coefficients

$$W_N \quad \text{with only one} \quad m_\alpha \neq 0$$

[Miller-Sahi]

Main results $SL(3), SL(4)$

[GKP14]

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More generally, for $\varphi \in \pi$

[GKP14]

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$SL(3), SL(4)$

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$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}} \qquad \text{where } \varphi_{\mathcal{O}} \text{ vanishes unless } \mathcal{O} \subseteq \mathrm{WF}(\pi)$$

[GKP14]

Main results $SL(3), SL(4)$

$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

[GKP14]

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Corollary

$\varphi \in \pi_{\min}$ maximally degenerate Whittaker coefficients

[GKP14]

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$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\min}$ maximally degenerate Whittaker coefficients single root

[GKP14]

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Corollary

$\varphi \in \pi_{\min}$

single root

[GKP14]

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$$\varphi = \sum_{\mathcal{O}} \varphi_{\mathcal{O}}$$ where $\varphi_{\mathcal{O}}$ vanishes unless $\mathcal{O} \subseteq \mathrm{WF}(\pi)$

Corollary

$\varphi \in \pi_{\min}$ single root

$\varphi \in \pi_{\mathrm{ntm}}$ at most two commuting roots

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

   $\pi_{\min}$

[GKP14]

Main results $SL(3), SL(4)$

Fourier coefficients on maximal parabolic subgroups in the minimal representation

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Theorem

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

[GKP14]

Main results

$SL(3), SL(4)$

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Maximal parabolic
Fourier coefficient

[GKP14]

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Known Whittaker coefficient

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Maximal parabolic
Fourier coefficient

Maximally degenerate

[GKP14]

Other groups

$$SL(n)$$

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic
Fourier coefficient

Maximally degenerate

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

$$SL(n)$$

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic Fourier coefficient

Maximally degenerate

and similar statement for next-to-minimal representation

[Work in progress with Ahlén, Liu, Kleinschmidt, Persson]

Other groups

Conjecture

A similar relations holds for all simple Lie groups

$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$ with $l \in L(\mathbb{Q})$ depending on ψ

Maximal parabolic
Fourier coefficient

Maximally degenerate

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Other groups

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A similar relations holds for all simple Lie groups

$$F_U(\chi_{\min}, \psi; g) = W_N(\chi_{\min}, \psi'; lg)$$ with $l \in L(\mathbb{Q})$ depending on ψ

Would allow us to compute non-perturbative effects that capture information about instantons and black holes

[GKP14]

[Proof in progress with Gourevitch, Kleinschmidt, Persson, Sahi]

Outlook

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- Other compactifications leading to automorphic forms on other groups.

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- Simplification of Fourier coefficients with $\chi_{\min}$ for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

  [Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

  How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

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- Other compactifications leading to automorphic forms on other groups.

- Simplification of Fourier coefficients with χ_{min} for dimensions lower than three. Kac-Moody groups E_9, E_{10}, E_{11}

  [Fleig-Kleinschmidt, Fleig-Kleinschmidt-Persson]

  How to define "small automorphic representations" for Kac-Moody groups? What is the mechanism behind the vanishing properties?

- $\mathcal{E}_6 D^6 R^4$ requires extended notion of automorphic forms, the development of which will positively bring new exciting insights to both physics and mathematics.

Thank you!

Henrik Gustafsson

hgustafsson.se